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Comodules and Landweber exact homology theories

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Abstract

We show that, if E is a commutative MU -algebra spectrum such that E_* is Landweber exact over MU_* , then the category of E_*E -comodules is equivalent to a localization of the category of MU_*MU -comodules. This localization depends only on the heights of E at the integer primes p . It follows, for example, that the category of $E(n)_*E(n)$ -comodules is equivalent to the category of $(v_n^{-1}BP)_*(v_n^{-1}BP)$ -comodules. These equivalences give simple proofs and generalizations of the Miller–Ravenel and Morava change of rings theorems. We also deduce structural results about the category of E_*E -comodules. We prove that every E_*E -comodule has a primitive, we give a classification of invariant prime ideals in E_* , and we give a version of the Landweber filtration theorem.

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0. Introduction

Suppose $E_*(-)$ and $R_*(-)$ are reduced homology theories with commutative products defined on finite CW complexes. Then $E_*(-)$ is said to be *Landweber exact*

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over $R_*(-)$ if there is a natural isomorphism

$$E_*(X) \cong E_* \otimes_{R_*} R_*(X)$$

for all finite CW complexes X . It then follows that this natural isomorphism extends to all CW complexes X , and indeed to all spectra X . Because of this, we usually just say that the spectrum E is Landweber exact over the spectrum R . Examples of this phenomenon abound in stable homotopy theory, and were first studied in some generality by Landweber [Lan76].

Example 0.1.

- (a) Conner and Floyd [CF66] showed that complex K -theory K is Landweber exact over complex cobordism MU , and also that real K -theory KO is Landweber exact over symplectic cobordism MSp . Hopkins and the first author [HH92] showed that KO is also Landweber exact over Spin cobordism $MSpin$.
- (b) The various elliptic cohomology theories [LRS95] are all Landweber exact over MU .
- (c) The Brown–Peterson spectrum BP at a prime p is Landweber exact over MU . Furthermore, the p -localization $MU_{(p)}$ of MU is also Landweber exact over BP [Lan76].
- (d) The Johnson–Wilson spectrum $E(n)$ [JW73] as well as the Morava E -theory spectrum E_n used in the work of Hopkins and Miller [HM] are Landweber exact over BP .
- (e) The Morava K -theory spectrum $K(n)$ is Landweber exact over the spectrum $P(n) = BPI_n$ [Yos76].

In all of the examples of spectra E above, the module E_*E is flat over E_* , so (E_*, E_*E) is a Hopf algebroid, or equivalently, a groupoid object in the opposite of the category of graded-commutative rings. For compatibility with the usual conventions in topology, we set up this correspondence so that the maps $\eta_L, \eta_R : E_* \rightarrow E_*E$ represent the maps sending a morphism to its target and source, respectively. We refer to [Rav86, Appendix 1] for basic facts about Hopf algebroids. The reduced homology E_*X is a comodule over the Hopf algebroid (E_*, E_*E) [Rav86, Proposition 2.2.8]. One of the main reasons this is important is because the E_2 -term of the Adams spectral sequence of X based on E is

$$\mathrm{Ext}_{E_*E}^{**}(E_*, E_*X)$$

and this Ext is taken in the category of E_*E -comodules. To help compute these E_2 -terms, various authors have constructed isomorphisms of the form $\mathrm{Ext}_\Gamma^{**}(M, N) \cong \mathrm{Ext}_\Sigma^{**}(M', N')$ under various hypotheses on the algebroids Γ and Σ , and the comodules M, N, M' and N' . This includes the change of rings theorems of Miller–Ravenel [MR77], Morava [Mor85], and the first author and Sadofsky [HS99]. The main result of this paper is that these isomorphisms come from equivalences of

comodule categories, and that such equivalences are much more common and systematic than was previously suspected.

The definition of Landweber exactness given above for homology theories has an analogue for Hopf algebroids. Given a Hopf algebroid (A, Γ) and an A -algebra B , we define B to be *Landweber exact* over (A, Γ) , or, by abuse of notation, over A , if the functor $B \otimes_A (-)$ is exact on the category of Γ -comodules. If $E_*(-)$ and $R_*(-)$ are homology theories as above, E is an R -module spectrum, and E_* is Landweber exact over R_* , then a well-known argument shows that E is Landweber exact over R . In the examples listed above, K_* , Ell_* , and BP_* are all Landweber exact over MU_* , and $(MU_*)_{(p)}$, $E(n)_*$, and E_{n*} are Landweber exact over BP_* , and $K(n)_*$ is Landweber exact over $P(n)_*$. On the other hand, it is not known whether KO_* is Landweber exact over MSp_* or $MSpin_*$.

In the above situation, it is important to understand the relationship between E_*E -comodules and R_*R -comodules. Given a Hopf algebroid (A, Γ) and a ring map $f: A \rightarrow B$, we put $\Gamma_B = B \otimes_A \Gamma \otimes_A B$. The pair (B, Γ_B) has a natural structure as a Hopf algebroid; we recall some details in Section 2. A central object of this paper is to make a detailed study of the relationship between the category of Γ_B -comodules and the category of Γ -comodules when B is Landweber exact over A . We prove the following theorem as Theorem 2.5.

Theorem A. *Suppose (A, Γ) is a flat Hopf algebroid and B is a Landweber exact A -algebra. Then the category of Γ_B -comodules is equivalent to the localization of the category of Γ -comodules with respect to the hereditary torsion theory*

$$\mathcal{T} = \{M \mid B \otimes_A M = 0\}.$$

To apply this to cases of interest in algebraic topology, we give a partial classification of graded hereditary torsion theories of BP_*BP -comodules in Theorem 3.1.

Theorem B. *Let \mathcal{T} be a graded hereditary torsion theory of BP_*BP -comodules, and suppose that \mathcal{T} contains a nonzero finitely presented comodule. Then either \mathcal{T} is the whole category of comodules, or there is an n such that \mathcal{T} is the collection of v_n -torsion comodules.*

This theorem is analogous to the classification of thick subcategories of finite p -local spectra [HS98].

This then leads to our main result, which is Theorem 4.2.

Theorem C. *Define the height of a Landweber exact BP_* -algebra E_* to be the largest n such that E_*/I_n is nonzero. If E_* and E'_* are Landweber exact BP_* -algebras of the same height, then the category of graded E_*E -comodules is equivalent to the category of graded E'_*E' -comodules. In particular, the categories of $E(n)_*E(n)$ -comodules, $E_{n*}E_{n*}$ -comodules, and $(v_n^{-1}BP)_*(v_n^{-1}BP)$ -comodules are all equivalent.*

As mentioned previously, this gives a simple explanation for the change of rings theorems of Miller–Ravenel, Morava, and Hovey–Sadofsky, all of which say that two Ext groups computed over different Hopf algebroids are isomorphic. Namely, the Ext groups are isomorphic because the categories they are computed in are equivalent.

When E_* is Landweber exact over BP_* , the category of E_*E -comodules is a localization of the category of BP_*BP -comodules, by Theorem A. This allows us to extend the standard structure theorems for BP_*BP -comodules of Landweber [Lan76] to E_*E -comodules. The following theorem is a summary of the results of Section 5.

Theorem D. *Suppose E_* is a Landweber exact BP_* -algebra of height $n \leq \infty$.*

- (a) *Every nonzero E_*E -comodule has a nonzero primitive.*
- (b) *If I is an invariant radical ideal in E_* , then $I = I_k$ for some $k \leq n$.*
- (c) *Every E_*E -comodule M that is finitely presented over E_* admits a finite filtration by subcomodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$$

for some s , with $M_r/M_{r-1} \cong s^t E_/I_j$ for some $j \leq n$ and some t , both depending on r .*

Remark. Baker [Bak95] has constructed a counterexample to a statement closely related to (a), in the case where E is the Morava E -theory spectrum E_n . This is not in fact a contradiction, because of the difference between $E_*E = \pi_*(E \wedge E)$ (which is used in our work) and $\pi_*L_{K(n)}(E \wedge E)$ (which is more closely related to the Morava stabiliser group, and is used in Baker's work). The topological comodule categories considered by Baker are well-known to be important, but they do not fit into our present framework; we hope to return to this in future.

The theorems we have just discussed all have global versions, where we replace BP_* by MU_* , and more local versions, where we replace BP_* by BP_*/J for a nice regular sequence J . We discuss these versions briefly at the end of the paper.

As mentioned above, the category of $E(n)_*E(n)$ -comodules is a localization of the category of BP_*BP -comodules. The resulting localization functor on BP_*BP -comodules is denoted L_n , and is analogous to the chromatic localization functor L_n much used in stable homotopy theory [Rav92]. The algebraic L_n is very interesting in its own right; it is left exact, and has interesting right derived functors L_n^i , which are closely related to local cohomology. The functor L_n and its derived functors are studied in [HS03].

We also point out that, to give a good algebraic model for stable homotopy theory, one wants a triangulated category rather than an abelian category. So there should be analogues of the theorems in this paper for some kind of derived categories of BP_*BP -comodules and E_*E -comodules. There are problems with the ordinary derived category; the first author has constructed a well-behaved replacement for it

in [Hov02b]. The authors have proved analogues of some of the theorems of this paper for these derived categories in [Hov02a].

1. Abelian localization

In this section, we summarize Gabriel’s theory of localization of abelian categories from an algebraic topologist’s point of view for the convenience of the reader. The original source for this material is [Gab62]; a standard source for localization in module categories is [Ste75]. The book [VOV79] gives a quick summary of the theory in an arbitrary Grothendieck category.

The following definition is standard in homotopy theory.

Definition 1.1. Suppose \mathcal{E} is a class of maps in a category \mathcal{C} . An object X of \mathcal{C} is said to be \mathcal{E} -local if $\mathcal{C}(f, X)$ is an isomorphism of sets for all $f \in \mathcal{E}$. We denote the full subcategory of \mathcal{E} -local objects by $L_{\mathcal{E}}\mathcal{C}$. An \mathcal{E} -localization of an object $M \in \mathcal{C}$ is a map $M \rightarrow LM$ in \mathcal{E} where $LM \in L_{\mathcal{E}}\mathcal{C}$. If every $M \in \mathcal{C}$ has an \mathcal{E} -localization, we say that \mathcal{E} -localizations exist.

It is also possible to define localizations without reference to the class \mathcal{E} .

Definition 1.2. A localization functor on a category \mathcal{C} is a functor $L: \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $\iota_M: M \rightarrow LM$ such that $L\iota_M = \iota_{LM}$ and this map is an isomorphism.

The following proposition is reasonably well-known; a version of it can be found in [HPS97, Section 3.1] and in other places.

Proposition 1.3. Suppose L is a localization functor on a category \mathcal{C} . Let \mathcal{E} denote the class of all maps f such that Lf is an isomorphism. Then ι_M is an \mathcal{E} -localization of M for all $M \in \mathcal{C}$. Conversely, if \mathcal{E} is a class of maps on \mathcal{C} such that an \mathcal{E} -localization $\iota_M: M \rightarrow LM$ exists for all $M \in \mathcal{C}$, then L is a localization functor. Furthermore, in either case L , thought of as a functor $L: \mathcal{C} \rightarrow L_{\mathcal{E}}\mathcal{C}$, is left adjoint to the inclusion functor.

We refer to the localization functor of Proposition 1.3 as localization with respect to \mathcal{E} .

A common way for localizations to arise is displayed in the following proposition.

Proposition 1.4. Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor with right adjoint G , and the counit of the adjunction $\varepsilon_M: FGM \rightarrow M$ is an isomorphism for all $M \in \mathcal{D}$. Then GF is the localization functor on \mathcal{C} with respect to $\mathcal{E} = \{f \mid Ff \text{ is an isomorphism}\}$. Furthermore, G defines an equivalence of categories $G: \mathcal{D} \rightarrow L_{\mathcal{E}}\mathcal{C}$.

Proof. Let $L = GF$. The natural transformation $\iota_M: M \rightarrow LM$ is the unit η_M of the adjunction. The two triangular relations of the adjunction say,

respectively, that

$$\varepsilon_{FM} \circ (F\eta_M) = 1_{FM} \quad \text{and} \quad (G\varepsilon_M) \circ \eta_{GM} = 1_{GM}.$$

In particular, $GF\eta_M = (G\varepsilon_{FM})^{-1} = \eta_{GFM}$. This means that $L\iota_M = \iota_{LM}$ and that this map is an isomorphism, as required. By Proposition 1.3, L is localization with respect to $\mathcal{E} = \{f \mid Lf \text{ is an isomorphism}\}$. Since FG is naturally isomorphic to the identity, one can easily check that GFf is an isomorphism if and only if Ff is an isomorphism.

Since FG is naturally isomorphic to the identity, G defines an equivalence of categories from \mathcal{D} to its image. Adjointness shows that GN is \mathcal{E} -local for all $N \in \mathcal{D}$. Conversely, the image of G contains LM for all $M \in \mathcal{C}$, so is a skeleton of $L_{\mathcal{E}}\mathcal{C}$. The result follows. \square

In point of fact, every localization functor arises from an adjunction as in Proposition 1.4; if L is a localization functor on \mathcal{C} , we can think of it as a functor $L: \mathcal{C} \rightarrow L_{\mathcal{E}}\mathcal{C}$, where it is left adjoint to the inclusion and satisfies the hypotheses of Proposition 1.4.

Now suppose that our category \mathcal{C} is abelian. It is natural, then, to consider localization functors arising from adjunctions $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ as in Proposition 1.4 where \mathcal{D} is also abelian and F is exact.

Definition 1.5. Suppose \mathcal{T} is a full subcategory of an abelian category \mathcal{C} . Then \mathcal{T} is called a *hereditary torsion theory* if \mathcal{T} is closed under subobjects, quotient objects, extensions, and arbitrary coproducts. When \mathcal{T} is a hereditary torsion theory, we define the class $\mathcal{E}_{\mathcal{T}}$ of \mathcal{T} -equivalences to consist of those maps whose cokernel and kernel are in \mathcal{T} . We define an object to be \mathcal{T} -local if and only if it is $\mathcal{E}_{\mathcal{T}}$ -local. We let $L_{\mathcal{T}}\mathcal{C}$ denote the full subcategory of \mathcal{T} -local objects.

Note that a hereditary torsion theory is just a Serre class that is closed under coproducts. Also, one can form the smallest hereditary torsion theory containing a specified class of objects by taking the intersection of all hereditary torsion theories containing that class.

Proposition 1.6. Suppose \mathcal{C} and \mathcal{D} are abelian categories, $F: \mathcal{C} \rightarrow \mathcal{D}$ is an exact functor with right adjoint G , and the counit of the adjunction $\varepsilon_M: FGM \rightarrow M$ is an isomorphism for all $M \in \mathcal{D}$. Then GF is the localization functor on \mathcal{C} with respect to the hereditary torsion theory $\mathcal{T} = \ker F = \{M \mid FM = 0\}$. Furthermore, G defines an equivalence of categories $G: \mathcal{D} \rightarrow L_{\mathcal{T}}\mathcal{C}$.

Proof. Proposition 1.4 implies that GF is localization with respect to

$$\mathcal{E} = \{f \mid Ff \text{ is an isomorphism}\}.$$

But, since F is exact, Ff is an isomorphism if and only if $F(\ker f) = F(\operatorname{coker} f) = 0$, which is true if and only if f is a \mathcal{T} -equivalence. \square

The main result of Gabriel on abelian localizations is the following theorem. Recall that a Grothendieck category is a cocomplete abelian category with a generator in which filtered colimits are exact.

Theorem 1.7. *Suppose \mathcal{T} is a hereditary torsion theory in a Grothendieck abelian category \mathcal{C} . Then \mathcal{T} -localizations exist.*

We outline the proof of Gabriel's Theorem 1.7, as we will need some of the ideas from this proof later. We first recall the characterization of \mathcal{T} -local objects.

Lemma 1.8. *Suppose \mathcal{T} is a hereditary torsion theory in an abelian category \mathcal{C} . An object X of \mathcal{C} is \mathcal{T} -local if and only if $\mathcal{C}(T, X) = \operatorname{Ext}_{\mathcal{C}}^1(T, X) = 0$ for all $T \in \mathcal{T}$.*

Recall that one can define Ext in an arbitrary abelian category without recourse to either projectives or injectives [ML95]. In particular, $\operatorname{Ext}_{\mathcal{C}}^1(M, N)$ is just the collection of all equivalence classes of short exact sequences

$$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0.$$

The usual exact sequences for Ext work in this generality.

Proof. Suppose first that X is \mathcal{T} -local, and $T \in \mathcal{T}$. Since $0 \rightarrow T$ is a \mathcal{T} -equivalence, we conclude that $\mathcal{C}(T, X) = 0$. Given a short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \rightarrow T \rightarrow 0,$$

the map f is a \mathcal{T} -equivalence, so $f^*: \mathcal{C}(Y, X) \rightarrow \mathcal{C}(X, X)$ is an isomorphism. Thus the identity map of X comes from a map $g: Y \rightarrow X$, and g defines a splitting of the given sequence. Hence $\operatorname{Ext}_{\mathcal{C}}^1(T, X) = 0$.

Conversely, suppose $\mathcal{C}(T, X) = \operatorname{Ext}_{\mathcal{C}}^1(T, X) = 0$ for all $T \in \mathcal{T}$, and $f: A \rightarrow B$ is a \mathcal{T} -equivalence. Consider the two short exact sequences

$$0 \rightarrow \ker f \rightarrow A \rightarrow \operatorname{im} f \rightarrow 0$$

and

$$0 \rightarrow \operatorname{im} f \rightarrow B \rightarrow \operatorname{coker} f \rightarrow 0.$$

By applying the functor $\mathcal{C}(-, X)$ to these short exact sequences, using the fact that

$$\mathcal{C}(\ker f, X) = \mathcal{C}(\operatorname{coker} f, X) = \operatorname{Ext}_{\mathcal{C}}^1(\operatorname{coker} f, X) = 0,$$

we see that $\mathcal{C}(f, X)$ is an isomorphism. \square

Now, in order to construct the localization $L_{\mathcal{T}}(X)$ of X with respect to a hereditary torsion theory \mathcal{T} in a Grothendieck category \mathcal{C} , we first form the union TX of all the subobjects of X that are in \mathcal{T} (these form a set because we are in a Grothendieck category). This gives us a \mathcal{T} -equivalence $X \rightarrow X/TX$. Then we taken an injective envelope I of X/TX (injective envelopes exist in a Grothendieck category), producing an exact sequence

$$0 \rightarrow X/TX \rightarrow I \rightarrow Q \rightarrow 0.$$

Finally, we let $L_{\mathcal{T}}(X)$ be the pullback $I \times_Q TQ$. The induced embedding $X/TX \rightarrow L_{\mathcal{T}}(X)$ is a \mathcal{T} -equivalence, and one can check that $L_{\mathcal{T}}(X)$ is \mathcal{T} -local.

Remark. In our case we will be working with graded abelian categories \mathcal{C} . This means that we have a given self-equivalence $s: \mathcal{C} \rightarrow \mathcal{C}$, which in fact is an isomorphism of categories in our examples. In this case, we define a full subcategory \mathcal{D} to be *graded* when $X \in \mathcal{D}$ if and only if $sX \in \mathcal{D}$. Similarly, a class of maps \mathcal{E} in \mathcal{C} is said to be *graded* when $f \in \mathcal{E}$ if and only if $sf \in \mathcal{E}$. The results of this section all have corresponding graded versions.

2. Landweber exact algebras

In this section, we apply localization techniques to comodules over Hopf algebroids. Recall that a Hopf algebroid is a pair of (possibly graded) commutative rings (A, Γ) so that $\text{Rings}(A, R)$ and $\text{Rings}(\Gamma, R)$ are the objects and morphisms of a groupoid that is natural in the (graded) commutative ring R . The precise structure maps and relations they satisfy can be found in [Rav86, Appendix 1]. The reason we are interested in them is that (E_*, E_*E) is a Hopf algebroid for many homology theories E , as explained in [Rav86, Proposition 2.2.8].

We will always assume our Hopf algebroids are *flat*; this means that the left unit $\eta_L: A \rightarrow \Gamma$ corepresenting the target of a morphism is a flat ring extension. This is the same as assuming that the right unit $\eta_R: A \rightarrow \Gamma$ corepresenting the source of a morphism is flat.

We note that in working with Hopf algebroids it is important to remember that $M \otimes_A N$ always denotes the tensor product of A -bimodules, using the right A -module structure on M and the left A -module structure on N . This mostly matters for Γ , which is a right A -module via the right unit η_R and a left A -module via the left unit η_L .

Recall that a *comodule* over a Hopf algebroid (A, Γ) is a left A -module M equipped with a coassociative and counital coaction map $\psi: M \rightarrow \Gamma \otimes_A M$. The category of Γ -comodules is abelian when (A, Γ) is flat [Rav86, Theorem A1.1.3].

We now recall the definition of Landweber exactness, mentioned in the introduction.

Definition 2.1. Suppose (A, Γ) is a flat Hopf algebroid, and $f: A \rightarrow B$ is a ring homomorphism. We say that B is *Landweber exact* over (A, Γ) , or just over A , if the functor $M \mapsto B \otimes_A M$ from Γ -comodules to B -modules is exact.

We next recall the construction of the Hopf algebroid Γ_B , and use it to reformulate the notion of Landweber exactness. The definition is motivated by the following construction on groupoids. Consider a groupoid with object set X and morphism set G . Given a set Y and a map $f: Y \rightarrow X$, we define a new groupoid (Y, G_f) as follows: the object set is Y , and the morphisms in G_f from y_1 to y_0 are the morphisms in G from $f(y_1)$ to $f(y_0)$, so as a set we have

$$G_f = Y \times_{X, f} G \times_{X, f} Y.$$

The map f induces a full and faithful functor $F: (Y, G_f) \rightarrow (X, G)$. To understand when this is an equivalence, consider the set

$$U = \{(y, g) \mid y \in Y, g \in G, f(y) = \text{target}(g)\} = Y \times_X G.$$

There is a map $\pi: U \rightarrow X$ given by $(y, g) \mapsto \text{source}(g)$. Our functor F is essentially surjective, and thus an equivalence, iff π is surjective.

Now suppose we have a Hopf algebroid (A, Γ) and an A -algebra B . For any ring T , we have a groupoid

$$(\text{Rings}(A, T), \text{Rings}(\Gamma, T))$$

and a map $\text{Rings}(B, T) \rightarrow \text{Rings}(A, T)$. We can apply the construction above to obtain a new groupoid $(\text{Rings}(B, T), \text{Rings}(\Gamma_B, T))$, where $\Gamma_B = B \otimes_A \Gamma \otimes_A B$ as before. The groupoid structure is natural in T , so Yoneda's lemma gives (B, Γ_B) the structure of a Hopf algebroid. For further details, see [Hov02c, p. 1315]. There is a morphism $\Phi = (f, \tilde{f}): (A, \Gamma) \rightarrow (B, \Gamma_B)$ of Hopf algebroids, where $\tilde{f}(u) = 1 \otimes u \otimes 1$; this corresponds to the functor F . The morphism Φ induces a functor Φ_* from Γ -comodules to Γ_B -comodules, given by $M \mapsto B \otimes_A M$; by definition, this is exact iff B is Landweber exact over A .

The map $\pi: U \rightarrow X$ corresponds to the ring map $f \otimes \eta_R: A \rightarrow B \otimes_A \Gamma$ given by $a \mapsto 1 \otimes \eta_R(a)$.

Lemma 2.2. Suppose (A, Γ) is a flat Hopf algebroid, and $f: A \rightarrow B$ is a ring homomorphism. Then B is Landweber exact if and only if the map $f \otimes \eta_R$ makes $B \otimes_A \Gamma$ into a flat A -algebra.

Proof. Suppose that B is Landweber exact, and $M \rightarrow N$ is a monomorphism of A -modules. Then $\Gamma \otimes_A f$ is a monomorphism as well, since Γ is flat as a right A -module. But $\Gamma \otimes_A M$ and $\Gamma \otimes_A N$ are both Γ -comodules, with the coaction coming from the diagonal on Γ . (This is called the *extended comodule structure* on $\Gamma \otimes_A M$). This makes $\Gamma \otimes_A f$ a comodule map. Since B is Landweber exact, we conclude that $B \otimes_A \Gamma \otimes_A f$ is a monomorphism.

Conversely, suppose that $B \otimes_A \Gamma$ is flat over A , and $u: M \rightarrow N$ is a monomorphism of comodules. The coaction map ψ_M is a split monomorphism of A -modules; the splitting is given by $\varepsilon \otimes 1$, where ε is the counit of (A, Γ) . Hence u is a retract of $\Gamma \otimes_A u$ as a map of A -modules. It follows that $B \otimes_A u$ is a retract of $B \otimes_A \Gamma \otimes_A u$ as a map of B -modules. Since $B \otimes_A \Gamma$ is flat over A , we conclude that $B \otimes_A u$ is a monomorphism, as required. \square

Corollary 2.3. *If B is Landweber exact over A , then Γ_B is flat over B (so the category of Γ_B -comodules is abelian).*

Proof. We have seen that $B \otimes_A \Gamma$ is flat as a right A -module; now take tensor products with B on the right. \square

For any morphism Φ of flat Hopf algebroids, the functor Φ_* obviously preserves colimits, so it should have a right adjoint Φ^* ; we next check that this works.

Lemma 2.4. *Suppose $\Phi: (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of flat Hopf algebroids. Then the functor $\Phi_*: \Gamma\text{-comod} \rightarrow \Sigma\text{-comod}$ defined by $\Phi_* M = B \otimes_A M$ has a right adjoint Φ^* .*

This lemma is proved in [Hov02b, Section 1], but it is central to our work, so we recall the proof here.

Proof. First note that any Σ -comodule N is the kernel of a map of extended comodules. Indeed, the structure map $\psi_N: N \rightarrow \Sigma \otimes_B N$ is a comodule map if we give $\Sigma \otimes_B N$ the extended comodule structure, and an embedding because it is split over B by the counit of Σ . If we let $q: \Sigma \otimes_B N \rightarrow Q$ denote the quotient, then we get a diagram of comodules

$$N \xrightarrow{\psi_N} \Sigma \otimes_B N \xrightarrow{\psi_Q q} \Sigma \otimes_B Q$$

expressing N as the kernel of a map of extended comodules. Adjointness forces us to define $\Phi^*(\Sigma \otimes_B P) = \Gamma \otimes_A P$ for any B -module P . Once we define Φ^* on maps between extended comodules such as $\psi_Q q$, we can then define $\Phi^* N$ as the kernel of $\Phi^*(\psi_Q q)$.

So suppose we have a map $f: \Sigma \otimes_B P \rightarrow \Sigma \otimes_B P'$. We need to define

$$\Phi^* f: \Gamma \otimes_A P \rightarrow \Gamma \otimes_A P'.$$

By adjointness, it suffices to define a map of A -modules $\Gamma \otimes_A P \rightarrow P'$. We define this map as the composite

$$\Gamma \otimes_A P \rightarrow \Sigma \otimes_B P \xrightarrow{f} \Sigma \otimes_B P' \rightarrow P'.$$

Here the first map is induced by the map $\Gamma \rightarrow \Sigma$ and the last map is induced by the counit of Σ . \square

Remark. It can be shown that when N is a Σ -comodule, the group $N \otimes_A \Gamma$ has compatible structures as a Γ -comodule and a Σ -comodule. This makes the Σ -primitives $\text{Prim}_\Sigma(N \otimes_A \Gamma)$ into a Γ -comodule, which turns out to be isomorphic to $\Phi^* N$. One can give a proof of the existence of Φ^* based on this formula, but we do not need it so we omit the details. Mark Behrens pointed out to the authors that

$$\Phi^* N \cong (\Gamma \otimes_A B) \square_\Sigma N,$$

where the symbol \square denotes the cotensor product; see [Rav86, Lemma A1.1.8].

We can now prove the main result of this section, which is also Theorem A of the introduction.

Theorem 2.5. *Suppose (A, Γ) is a flat Hopf algebroid, and B is a Landweber exact A -algebra. Let $\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$ denote the corresponding map of Hopf algebroids, inducing $\Phi_*: \Gamma\text{-comod} \rightarrow \Gamma_B\text{-comod}$ with right adjoint Φ^* . Then the counit of the adjunction $\varepsilon: \Phi_* \Phi^* M \rightarrow M$ is a natural isomorphism. Hence $\Phi^* \Phi_*$ is localization with respect to the hereditary torsion theory $\mathcal{T} = \ker \Phi_*$, and Φ^* defines an equivalence of categories from $\Gamma_B\text{-comod}$ to $L_{\mathcal{T}}(\Gamma\text{-comod})$.*

Proof. Since B is Landweber exact, Φ_* is exact, so ε is a natural transformation of left exact functors. Since every comodule is a kernel of a map of extended comodules, it suffices to check that ε_N is an isomorphism for extended comodules $N = \Gamma_B \otimes_B V$. But then we have

$$\begin{aligned} \Phi_* \Phi^* N &\cong B \otimes_A (\Gamma \otimes_A V) \cong B \otimes_A \Gamma \otimes_A B \otimes_B V \\ &\cong \Gamma_B \otimes_B V \cong M, \end{aligned}$$

as required. Proposition 1.6 completes the proof. \square

3. Torsion theories of BP_*BP -comodules

Suppose (A, Γ) is a flat Hopf algebroid, and B is a Landweber exact A -algebra. Theorem 2.5 shows that the category of Γ_B -comodules is equivalent to the localization of the category of Γ -comodules with respect to some hereditary torsion theory \mathcal{T} . Thus we would like to classify all hereditary torsion theories of Γ -comodules. This is of course impossible in general, but it turns out to be tractable in the cases of most interest in algebraic topology. In this section, we concentrate on the case $(A, \Gamma) = (BP_*, BP_*BP)$, where BP is the Brown–Peterson spectrum.

Recall that $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where $|v_i| = 2(p^i - 1)$. We choose the v_i to be the Araki generators [Rav86, Section A2.2] for definiteness, but all that matters is that v_n

is primitive modulo $I_n = (p, v_1, \dots, v_{n-1})$. The ideals I_n are independent of the choice of generators. For notational purposes, we take $v_0 = p$ and $v_{-1} = 0$. We also write s for the shift functor on BP_*BP -comodules, so that $(sM)_n = M_{n-1}$.

Let \mathcal{T}_n denote the class of all graded BP_*BP -comodules that are v_n -torsion, in the sense that each element is killed by some power of v_n , depending on the element. By Lemma 2.3 of [JY80], M is v_n -torsion if and only if M is I_{n+1} -torsion, so \mathcal{T}_n is independent of the choice of generators.

The following theorem is Theorem B of the introduction.

Theorem 3.1. *Let \mathcal{T} be a graded hereditary torsion theory of graded BP_*BP -comodules, and suppose that \mathcal{T} contains some nonzero comodule that is finitely presented over BP_* . Then $\mathcal{T} = \mathcal{T}_n$ for some $n \geq -1$.*

The reader should compare Theorem 3.1 to the classification of Serre classes of finitely presented BP_*BP -comodules in [JLR96] (which they call thick subcategories). Given a hereditary torsion theory \mathcal{T} , the collection \mathcal{F} of all finitely presented comodules in it is a Serre class (of all the finitely presented comodules); combining Theorem 3.1 with the result of [JLR96] says that as long as \mathcal{F} is nonzero, then \mathcal{T} is uniquely determined by \mathcal{F} .

We do not know what happens when there are no nonzero finitely presented comodules in \mathcal{T} . In this case, Proposition 3.3 below implies that every comodule in \mathcal{T} is v_n -torsion for all n . Ravenel [Rav84, Section 2] conjectures that there are uncountably many different Bousfield classes of spectra BPI where I is an infinite regular sequence in BP_* . One might similarly conjecture that there are uncountably many different hereditary torsion theories \mathcal{T} containing no nonzero finitely presented comodules.

Theorem 3.1 will follow from the two propositions below.

Proposition 3.2. *\mathcal{T}_n is the graded hereditary torsion theory generated by the BP_*BP -comodule BP_*/I_{n+1} .*

Proposition 3.3. *Suppose that \mathcal{T} is a graded hereditary torsion theory of graded BP_*BP -comodules such that $BP_*/I_n \notin \mathcal{T}$. Then $\mathcal{T} \subseteq \mathcal{T}_n$.*

Given these two propositions, Theorem 3.1 follows easily.

Proof of Theorem 3.1. Suppose \mathcal{T} is a graded hereditary torsion theory containing the nonzero finitely presented comodule M . The Landweber filtration theorem for BP_*BP -comodules [Lan76, Theorem 2.3] guarantees that M has a subcomodule of the form $s^t BP_*/I_j$ for some j and some t . Thus $BP_*/I_j \in \mathcal{T}$. Let

$$n+1 = \min\{j \mid BP_*/I_j \in \mathcal{T}\}.$$

Then $\mathcal{T} \supseteq \mathcal{T}_n$ by Proposition 3.2. On the other hand, $BP_*/I_n \notin \mathcal{T}$, so $\mathcal{T} \subseteq \mathcal{T}_n$ by Proposition 3.3. Hence $\mathcal{T} = \mathcal{T}_n$, as required. \square

We owe the reader proofs of Propositions 3.2 and 3.3. We need the following lemma.

Lemma 3.4. *Suppose M is a nonzero v_n -torsion graded BP_*BP -comodule. Then M has a nonzero primitive x such that $I_{n+1} \subseteq \text{Ann}(x)$.*

Proof. Let y be a nonzero element of M , and let $I = \sqrt{\text{Ann } y}$. Since y is v_n -torsion, it is also I_{n+1} -torsion, and so $I_{n+1} \subseteq I$. Theorem 2 of [Lan79] guarantees that there is a primitive x with $\text{Ann}(x) = I$. \square

Proof of Proposition 3.2. Let \mathcal{T} denote the graded hereditary torsion theory generated by BP_*/I_{n+1} . Since one can easily check that \mathcal{T}_n is a graded hereditary torsion theory, and $BP_*/I_{n+1} \in \mathcal{T}_n$, we see that $\mathcal{T} \subseteq \mathcal{T}_n$. Conversely, suppose M is v_n -torsion. We construct a transfinite increasing sequence M_α of subcomodules of M such that each M_α is in \mathcal{T} . This sequence will be strictly increasing unless $M_\beta = M$ for some β , so we conclude that $M = M_\beta \in \mathcal{T}$.

To construct M_0 , we use Lemma 3.4 to find a nonzero primitive $x \in M$ such that $I_{n+1}x = 0$. This gives a subcomodule $M_0 \cong s^t BP_*/I$ of M such that $I \supseteq I_{n+1}$. Hence M_0 is isomorphic to a quotient of $s^t BP_*/I_{n+1}$, so $M_0 \in \mathcal{T}$. This begins the transfinite induction. The limit ordinal step of the induction is simple. If we have defined M_α for all $\alpha < \beta$ for some limit ordinal β , we define $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$. Since M_β is a quotient of $\bigoplus_{\alpha < \beta} M_\alpha$, M_β is still in \mathcal{T} .

We now carry out the successor ordinal step of the induction. So suppose we have defined M_α . If $M_\alpha = M$, we let $M_{\alpha+1} = M$ as well. Otherwise, consider the quotient M/M_α . Since this comodule is v_n -torsion, Lemma 3.4 gives us an element $z \in M$ such that the coset \bar{z} in M/M_α is a nonzero primitive such that $I_{n+1}\bar{z} = 0$. We define $M_{\alpha+1}$ to be the subcomodule of M generated by M_α and z . Then $M_{\alpha+1}$ is an extension of M_α and $s^t BP_*/\text{Ann}(\bar{z})$ for some t , so $M_{\alpha+1} \in \mathcal{T}$ as required. This completes the proof. \square

Proof of Proposition 3.3. Suppose BP_*/I_n is not in some graded hereditary torsion theory \mathcal{T} , and M is in \mathcal{T} . We must show that every $x \in M$ is v_n -torsion. We show that every $x \in M$ is v_j -torsion for $-1 \leq j \leq n$ by induction on j . The initial step is automatic since $v_{-1} = 0$. For the induction step, assume that we have shown that M is v_{j-1} -torsion, and let M_j denote the v_j -torsion in M . This is a subcomodule of M by Proposition 2.9 of [JY80]. Suppose that M_j is not all of M . Lemma 3.4 allows us to find a primitive y_j in M/M_j with $I_j \subseteq \text{Ann}(y_j)$. Since M/M_j is v_j -torsion-free, and the only primitives modulo I_j are powers of v_j ([Rav86, Theorem 4.3.2]), we conclude that $\text{Ann}(y_j) = I_j$. This gives us an embedding $s^t BP_*/I_j \subseteq M/M_j$, so $BP_*/I_j \in \mathcal{T}$. This contradiction shows that M is all v_j -torsion, as required. \square

4. Equivalences of comodule categories

In this section, we show that the category of comodules over a Landweber exact BP_* -algebra B depends only on the height of B , and deduce versions of the Miller–Ravenel and Morava change of rings theorems.

Definition 4.1. Suppose B is a nonzero graded BP_* -module. We define the *height* of B , written $\text{ht } B$, to be the largest n such that B/I_n is nonzero, or ∞ if B/I_n is nonzero for all n .

Note that $\text{ht } E(n)_* = \text{ht } v_n^{-1}BP_* = n$ and $\text{ht } BP_* = \infty$. The following theorem implies Theorem C of the introduction.

Theorem 4.2. Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B and B' are two graded Landweber exact BP_* -algebras with $\text{ht } B = \text{ht } B' = n \leq \infty$. Then the category of graded Γ_B -comodules is equivalent to the category of graded $\Gamma_{B'}$ -comodules. If $n = \infty$, these categories are equivalent to the category of graded Γ -comodules. If $n < \infty$, these categories are equivalent to the localization of the category of graded Γ -comodules with respect to the graded hereditary torsion theory \mathcal{T}_n .

Proof. Theorem 2.5 implies that the category of graded Γ_B -comodules is equivalent to the localization of the category of graded Γ -comodules with respect to the kernel \mathcal{T} of the functor $M \mapsto B \otimes_A M$. Suppose first that $n < \infty$. Then $BP_*/I_n \notin \mathcal{T}$ but $BP_*/I_{n+1} \in \mathcal{T}$. Theorem 3.1, or, more precisely, Propositions 3.2 and 3.3, imply that $\mathcal{T} = \mathcal{T}_n$.

Now suppose that $n = \infty$. We claim that $\mathcal{T} = (0)$. Indeed, suppose $M \in \mathcal{T}$ and is nonzero. Since every graded BP_*BP -comodule has a primitive, M has a subcomodule isomorphic to $s^t BP_*/I$ for some invariant ideal I . But BP_* is a local ring (in the graded sense), with maximal ideal $I_\infty = \bigcup_r I_r$. Thus $I \subseteq I_\infty$, and so $BP_*/I_\infty \in \mathcal{T}$, since it is a quotient of BP_*/I . Hence $B/(I_\infty B) = 0$, and so the unit $1 \in B$ is in $I_\infty B$, so must be in $I_r B$ for some r . Hence $B/I_r B = 0$, contradicting our assumption that B has infinite height. Thus $\mathcal{T} = (0)$. \square

In view of this theorem, we denote the localization functor on BP_*BP -comodules corresponding to the torsion theory \mathcal{T}_n by $L_n : BP_*BP\text{-comod} \rightarrow BP_*BP\text{-comod}$.

Corollary 4.3. Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B is a Landweber exact BP_* -module of finite height n . Then the category of graded Γ_B -comodules is equivalent to the full subcategory of graded Γ -comodules M such that

$$\text{Hom}_A^*(A/I_{n+1}, M) = \text{Ext}_\Gamma^{1,*}(A/I_{n+1}, M) = 0.$$

Proof. It suffices to prove that M is L_n -local if and only if M satisfies the given conditions. Lemma 1.8 tells us that M is local if and only if

$$\mathrm{Hom}_\Gamma^*(T, M) = \mathrm{Ext}_\Gamma^{1,*}(T, M) = 0$$

for all v_n -torsion comodules T .

We first claim that $\mathrm{Hom}_\Gamma^*(T, M) = 0$ for all v_n -torsion comodules T if and only if $\mathrm{Hom}_A^*(A/I_{n+1}, M) = 0$. To see this, note that $\mathrm{Hom}_A^*(A/I_{n+1}, M) = 0$ if and only if M has no nonzero v_n -torsion elements (using the equivalence of v_n -torsion with I_{n+1} -torsion [JY80, Lemma 2.3]). But then $\mathrm{Hom}_A^*(T, M) = 0$ for all v_n -torsion modules T , and so, a fortiori, $\mathrm{Hom}_\Gamma^*(T, M) = 0$ for all v_n -torsion comodules T . Conversely, if $\mathrm{Hom}_\Gamma^*(A/I_{n+1}, M) = 0$, then M has no primitives that are v_n -torsion. Proposition 2.7 of [JY80] implies that M has no v_n -torsion at all, and so $\mathrm{Hom}_A^*(A/I_{n+1}, M) = 0$ as well.

We now claim that $\mathrm{Ext}_\Gamma^{1,*}(A/I_{n+1}, M) = 0$ if and only if $\mathrm{Ext}_\Gamma^{1,*}(T, M) = 0$ for all v_n -torsion comodules T , as long as $\mathrm{Hom}_A^*(A/I_{n+1}, M) = 0$. Indeed, suppose $\mathrm{Ext}_\Gamma^{1,*}(A/I_{n+1}, M) = 0$. Using the short exact sequences

$$0 \rightarrow A/I_{n+j} \xrightarrow{v_{n+j}} A/I_{n+j} \rightarrow A/I_{n+j+1} \rightarrow 0,$$

we conclude that $\mathrm{Ext}_\Gamma^{1,*}(A/I_r, M) = 0$ for all $r > n$. The Landweber filtration theorem [Lan76, Theorem 2.3] tells us that every finitely presented Γ -comodule T is an iterated extension of suspensions of comodules of the form A/I_r ; if T is v_n -torsion, one can easily check that $r > n$. Thus $\mathrm{Ext}_\Gamma^1(T, M) = 0$ for all finitely presented v_n -torsion comodules. Now every comodule T is a filtered colimit of finitely presented comodules, by Lemma 1.15 of [JY80]. If T is v_n -torsion, then the finitely presented v_n -torsion comodules mapping to T are cofinal, so T is a filtered colimit $\mathrm{colim} T_\alpha$ of finitely presented v_n -torsion comodules. This gives a short exact sequence $T'' \rightarrow T' \rightarrow T$, where $T' = \bigoplus_\alpha T_\alpha$ (so $\mathrm{Ext}_\Gamma^{1,*}(T', M) = 0$) and $T'' \leq T'$ (so T'' is v_n -torsion and thus $\mathrm{Hom}_\Gamma^*(T'', M) = 0$). It follows that $\mathrm{Ext}_\Gamma^{1,*}(T, M) = 0$ as required. \square

In [HS03], we strengthen Corollary 4.3 by showing that a BP_*BP -comodule M is L_n -local if and only if

$$\mathrm{Hom}_{BP_*}^*(BP_*/I_{n+1}, M) = \mathrm{Ext}_{BP_*}^{1,*}(BP_*/I_{n+1}, M) = 0,$$

with the Ext group computed over the ring BP_* rather than the Hopf algebroid BP_*BP . This Ext group can easily be computed from a Koszul resolution and so is much more accessible than the previous one.

Corollary 4.4. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose $B \rightarrow B'$ is a map of Landweber exact A -algebras. Let \mathcal{T} denote the graded hereditary torsion theory of Γ_B -comodules generated by $B/I_{\mathrm{ht} B'+1}$ if $\mathrm{ht} B' < \infty$ and (0) if $\mathrm{ht} B' = \infty$. Then the*

functor $M \mapsto B' \otimes_B M$ defines an equivalence of categories between the localization of the category of graded Γ_B -comodules with respect to \mathcal{T} and the category of graded $\Gamma_{B'}$ -comodules. In particular, if $\text{ht } B = \text{ht } B'$, then this functor is itself an equivalence of categories.

This corollary is a special case of the general fact that maps between localizations are themselves localizations; see [HPS97, Lemma 3.1.5].

Example 4.5. There are well-known maps

$$v_n^{-1}BP_* \rightarrow E(n)_* \rightarrow E_{n*}$$

of Landweber exact BP_* -algebras of height n . These maps induce equivalences of the associated categories of comodules. Note that they certainly do not induce equivalences of the associated categories of modules; in particular, $E(n)_*$ is Noetherian and $v_n^{-1}BP_*$ is not.

We can now give a straightforward and systematic account of some well-known change of rings theorems, as mentioned in the introduction. The following is our general result; it follows immediately from Corollary 4.4.

Theorem 4.6. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose $B \rightarrow B'$ is a map of Landweber exact A -algebras such that $\text{ht } B = \text{ht } B'$. Then*

$$\text{Ext}_{\Gamma_B}^{**}(M, N) \cong \text{Ext}_{\Gamma_{B'}}^{**}(B' \otimes_B M, B' \otimes_B N).$$

The Morava change of rings theorem [Mor85] is often stated in precisely this form. We give a graded version of it, as opposed to the ungraded version given in [Rav86, Theorem 6.1.3].

Corollary 4.7. *Suppose $(A, \Gamma) = (BP_*, BP_*BP)$, and let I denote the ideal in A generated by p and all the v_i except v_n . Let B denote the completion of $v_n^{-1}A$ at I , and let B' denote the completion of $E(n)_*$ at I_n . Then*

$$\text{Ext}_{\Gamma_B}^{**}(M, N) \cong \text{Ext}_{\Gamma_{B'}}^{**}(B' \otimes_B M, B' \otimes_B N)$$

for all Γ_B -comodules M and N .

Note that the Morava change of rings theorem was only known before in case $M = B$.

Here is our version of the Miller–Ravenel change of rings theorem [MR77, Theorem 3.10].

Corollary 4.8. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, $B = v_n^{-1}BP_*$, and $B' = E(n)_*$. Then*

$$\mathrm{Ext}_{\Gamma_B}^{**}(M, N) \cong \mathrm{Ext}_{\Gamma_{B'}}^{**}(B' \otimes_B M, B' \otimes_B N)$$

for all Γ_B -comodules M and N .

The Miller–Ravenel change of rings theorem is usually stated as

$$\mathrm{Ext}_{BP_*BP}^{**}(BP_*, N) \cong \mathrm{Ext}_{E(n)_*E(n)}^{**}(E(n)_*, E(n)_* \otimes_{BP_*} N) \quad (4.9)$$

for all BP_* -comodules N on which v_n acts invertibly. This is a consequence of Corollary 4.8, arguing as in Lemmas 3.11 and 3.12 of [MR77]. The point is essentially as follows: if v_n acts invertibly on N , then nothing changes if we invert v_n in BP_* and BP_*BP . Moreover, N is necessarily I_n -torsion, so v_n is asymptotically primitive on anything involving N , so we need not distinguish between inverting $\eta_L(v_n)$ or $\eta_R(v_n)$.

We have also generalized the statement 4.9 of the Miller–Ravenel change of rings theorem in [Hov02a], expressing it as an isomorphism between morphism sets in appropriate derived categories.

Similarly, if $m \geq n$, we can apply Theorem 4.6 to the map $v_n^{-1}BP_* \rightarrow v_n^{-1}E(m)_*$ to get a version of the change of rings theorem of [HS99, Theorem 3.1].

Corollary 4.10. *Suppose $(A, \Gamma) = (BP_*, BP_*BP)$, and let $B = v_n^{-1}BP_*$ and $B' = v_n^{-1}E(m)_*$ for $m \geq n$. Then*

$$\mathrm{Ext}_{\Gamma_B}^{**}(M, N) \cong \mathrm{Ext}_{\Gamma_{B'}}^{**}(B' \otimes_B M, B' \otimes_B N)$$

for all Γ_B -comodules M and N .

Again, the methods of Miller and Ravenel allow one to derive the original change of rings theorem of the first author and Sadofsky from Corollary 4.10.

5. The structure of E_*E -comodules

This section is devoted to proving analogues for E_*E -comodules of the Landweber structure theorems for BP_*BP -comodules, when E_* is Landweber exact over BP_* .

Theorem 5.1. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B is a Landweber exact A -algebra. Then every nonzero Γ_B -comodule has a nonzero primitive.*

Proof. Let $\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$ denote the evident map of Hopf algebroids. Let $\Phi_*: \Gamma\text{-comod} \rightarrow \Gamma_B\text{-comod}$ denote the induced functor, with right adjoint Φ^* . Suppose M is a nonzero Γ_B -comodule. We must show that $\mathrm{Hom}_{\Gamma_B}^*(B, M)$ is

nonzero. But adjointness implies that

$$\mathrm{Hom}_{\Gamma_B}^*(B, M) = \mathrm{Hom}_{\Gamma_B}^*(\Phi_* A, M) \cong \mathrm{Hom}_{\Gamma}^*(A, \Phi^* M).$$

Since $\Phi_* \Phi^* M \cong M$ by Theorem 2.5, $\Phi^* M$ is a nonzero Γ -comodule. It is well-known that every nonzero Γ -comodule has a primitive; it follows for example from Lemma 3.4. Thus $\mathrm{Hom}_{\Gamma}^*(A, \Phi^* M)$ is nonzero, as required. \square

We now compute the primitives in B/I_n for all n . The case $B = BP_*$ is well-known [Rav86, Theorem 4.3.2].

Theorem 5.2. *Let $(A, \Gamma) = (BP_*, BP_* BP)$, and suppose B is a nonzero Landweber exact A -algebra.*

- (a) *If $\mathrm{ht} B > 0$, then $\mathrm{Hom}_{\Gamma_B}^*(B, B) \cong \mathbb{Z}_{(p)}$.*
- (b) *If $\mathrm{ht} B = 0$, then $\mathrm{Hom}_{\Gamma_B}^*(B, B) \cong \mathbb{Q}$.*
- (c) *If $\mathrm{ht} B > n > 0$, then $\mathrm{Hom}_{\Gamma_B}^*(B, B/I_n) \cong \mathbb{F}_p[v_n]$.*
- (d) *If $\infty > \mathrm{ht} B = n > 0$, then $\mathrm{Hom}_{\Gamma_B}^*(B, B/I_n) \cong \mathbb{F}_p[v_n, v_n^{-1}]$.*
- (e) *If $\mathrm{ht} B = \infty$ then $\mathrm{Hom}_{\Gamma_B}^*(B, B/I_\infty) \cong \mathbb{F}_p$.*
- (f) *If $n > \mathrm{ht} B$ then $B/I_n = 0$ and so $\mathrm{Hom}_{\Gamma_B}^*(B, B/I_n) = 0$.*

The simplest way to prove this theorem is to use the following computation. Recall that L_n denotes the localization functor on the category of $BP_* BP$ -comodules with respect to the hereditary torsion theory of v_n -torsion comodules.

Lemma 5.3. *For $n < \infty$ we have*

$$L_n(BP_*/I_n) \cong v_n^{-1} BP_*/I_n$$

and

$$L_n(BP_*/I_m) = BP_*/I_m$$

for $m < n$.

As usual, we let $v_0 = p$ and $I_0 = (0)$ in interpreting the statement of this lemma. Recall also that L_∞ is the identity functor, so the $n = \infty$ case is trivial.

Proof. Let M denote either BP_*/I_m (for $m < n$) or $v_n^{-1} BP_*/I_n$ (for $m = n$). It suffices to show that M is L_n -local, since the map $BP_*/I_n \rightarrow v_n^{-1} BP_*/I_n$ has v_n -torsion cokernel, so is an L_n -equivalence. For this, we use Corollary 4.3. Since M has no v_n -torsion, it suffices to show that $\mathrm{Ext}_{BP_* BP}^{1,*}(BP_*/I_{n+1}, M) = 0$.

We first show that $\mathrm{Ext}_{BP_*}^{1,*}(BP_*/I_{n+1}, M) = 0$. So suppose we have a short exact sequence of BP_* -modules

$$0 \rightarrow M \xrightarrow{i} X \xrightarrow{g} s^t BP_*/I_{n+1} \rightarrow 0. \quad (5.4)$$

Let $x \in X$ be such that $g(x)$ is the generator of $s^t BP_*/I_{n+1}$. The argument now depends on whether $M = BP_*/I_m$ or $M = v_n^{-1} BP_*/I_n$. In case $M = v_n^{-1} BP_*/I_n$, note that $v_n x \in M$. Since v_n acts invertibly on M , there is a $y \in M$ such that $v_n y = v_n x$. Then $v_n(x - y) = 0$, and also $v_i(x - y) \in M$ for $i < n$. Hence $v_n v_i(x - y) = 0$, and so $v_i(x - y) = 0$ since M has no v_n -torsion. Thus $x - y$ defines a splitting of our given exact sequence (5.4).

Now suppose $M = BP_*/I_m$ for some $m < n$. Then $v_m x$ and $v_{m+1} x$ are both in M . Since $v_{m+1}(v_m x) = v_m(v_{m+1} x)$ and M is a unique factorization domain, we conclude that $v_m x = v_m y$ for some $y \in M$, and that $v_{m+1} x = v_{m+1} y$. Now a similar argument as we used in case $M = v_n^{-1} BP_*/I_n$ shows that $v_i x = v_i y$ for all $i \leq n$. Hence $x - y$ defines a splitting of (5.4).

Now suppose the short exact sequence (5.4) is a sequence of $BP_* BP$ -comodules. Write $X \cong M \oplus s^t BP_*/I_{n+1}$ as BP_* -modules. We claim that this splitting must be a splitting of $BP_* BP$ -comodules as well. Indeed, let Y be the v_n -torsion in X , which is a subcomodule and is obviously just $0 \oplus s^t BP_*/I_{n+1}$. Hence the map $Y \rightarrow X \rightarrow BP_*/I_{n+1}$ is an isomorphism, and its inverse gives a splitting of the sequence. \square

Proof of Theorem 5.2. As usual, let Φ_* denote the functor from Γ -comodules to Γ_B -comodules that takes M to $B \otimes_A M$. Then we have

$$\begin{aligned} \operatorname{Hom}_{\Gamma_B}^*(B, B/I_m) &= \operatorname{Hom}_{\Gamma_B}^*(\Phi_* A, \Phi_* A/I_m) \\ &\cong \operatorname{Hom}_{\Gamma}^*(A, \Phi^* \Phi_*(A/I_m)) \cong \operatorname{Hom}_{\Gamma}^*(A, L_n(A/I_m)). \end{aligned}$$

Parts (a)–(d) of the theorem now follow from Lemma 5.3 and [Rav86, Proposition 5.1.12]. Part (e) follows from the observation that $\operatorname{Hom}_{\Gamma}^*(A, BP_*/I_{\infty}) = BP_*/I_{\infty} = \mathbb{F}_p$. Part (f) is just the definition of $\operatorname{ht} B$, recorded for ease of comparison. \square

We now consider the analogue of Landweber's classification of invariant radical ideals in BP_* [Lan76, Theorem 2.2]. For this to work smoothly, we need to modify the problem slightly. Consider an abelian category \mathcal{A} with a symmetric monoidal tensor product, and let k be the unit for the tensor product. We define a *categorical ideal* in \mathcal{A} to be a subobject of k ; given ideals I and J , we let IJ denote the image of the evident map $I \otimes J \rightarrow k$. We say that I is *categorically radical* if $J^2 \leq I$ implies $J \leq I$. This notion is evidently invariant under monoidal equivalences of abelian categories, such as those in Theorem 4.2.

We now specialize to the case $\mathcal{A} = (B, \Sigma)$ -comod. The categorical ideals are then the invariant ideals in B . An invariant radical ideal is categorically radical, but the converse need not be true. For example, take $(A, \Gamma) = (BP_*, BP_* BP)$ as before, and $B = BP_*[x]/x^2$, and $\Sigma = \Gamma_B$. Then $I_n B$ is categorically radical, but not radical. Indeed, the only invariant ideals are those of the form IB for some invariant ideal $I \leq BP_*$, and IB is never radical.

One can easily check that the proof of Landweber's classification of invariant radical ideals in BP_* in [Rav86, Theorem 4.3.2] also classifies categorically radical ideals. That is, we have the following theorem.

Theorem 5.5. *The ideal $I \leq BP_*$ is a categorically radical invariant ideal if and only if $I = I_n$ for some $0 \leq n \leq \infty$.*

Almost the same theorem holds for categorically radical ideals in Landweber exact BP_* -algebras.

Theorem 5.6. *Suppose $(A, \Gamma) = (BP_*, BP_*BP)$ and B is a Landweber exact A -algebra. Then the categorically radical invariant ideals in B are $\{I_k B \mid 0 \leq k \leq \text{ht } B\}$. In particular, this set contains all the invariant radical ideals.*

Proof. Put $n = \text{ht } B$. We assume that $n > 0$, leaving the rational case to the reader. As $n > 0$, we have $\Phi^* B = L_n BP_* = BP_* = A$. Note also that Φ^* is left exact, so it preserves monomorphisms, so it sends invariant ideals in B to invariant ideals in A . Consider a categorically radical invariant ideal $J \leq B$, and put $K = \Phi^* J \leq A$, so $J = \Phi_* K = KB$. If $K = A$ this means that $J = B$, which we have implicitly excluded from consideration; so K is a proper ideal in A . We claim that K is also categorically radical. Indeed, suppose we have an invariant ideal K_0 with $K_0^2 \leq K$. Put $J_0 = BK_0$ and apply Φ_* to the maps $K_0 \otimes K_0 \rightarrow K \rightarrow A$ to see that $J_0^2 \leq J$. As J is categorically radical, we have $J_0 \leq J$, so $L_n K_0 = \Phi^* \Phi_* K_0 = \Phi^* J_0 \leq \Phi^* J = K$. Moreover, as A is an integral domain, K_0 has no I_{n+1} -torsion, so $K_0 \leq L_n K_0$, so $K_0 \leq K$ as required.

Since K is categorically radical, we must have $K = I_k$ for some $0 \leq k \leq \infty$. If $k > \text{ht } B$, then $J = \Phi_* K = B$. Hence $0 \leq k \leq \text{ht } B$. \square

This classification leads to the analogue of the Landweber filtration theorem, proved by Landweber [Lan76] for BP_*BP -comodules.

Theorem 5.7. *Suppose $(A, \Gamma) = (BP_*, BP_*BP)$, and let B be a Landweber exact A -algebra. Then every Γ_B -comodule M that is finitely presented over B admits a finite filtration by subcomodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$$

for some s , with $M_r/M_{r-1} \cong s^{t_r} B/I_{j_r}$ with $j_r \leq \text{ht } B$ for all r .

Proof. First note that M is finitely presented over B if and only if M is a finitely presented object of $\Gamma_B\text{-comod}$; that is, if and only if $\text{Hom}_{\Gamma_B}^*(M, -)$ commutes with all filtered colimits. This is proved in [Hov02b, Proposition 1.3.3]. It follows that the statement of Theorem 5.7 is invariant under the equivalences of categories in Theorem 4.2. Hence we can assume that either $B = BP_*$ or $B = E(n)_*$. The case $B = BP_*$ is the classical Landweber filtration theorem.

So now suppose $B = E(n)_*$ and M is an arbitrary graded Γ_B -comodule. We construct a subcomodule of M isomorphic to $s^t B/I_m$ for some p and some $m \leq n$. Indeed, choose a nonzero primitive y_0 in M . If $\text{Ann}(y_0) = (0)$, we are done. If not, $p^i y_0 = 0$ for some i by Theorem 5.2. Choose a minimal such i and let $y_1 = p^{i-1} y_0$.

Then $\text{Ann}(y_1)$ is a proper invariant ideal containing (p) . If it is (p) , we are done. If not, then Theorem 5.2 implies that $v_1^j y_1 = 0$ for some minimal $j > 0$. Let $y_2 = v_1^{j-1} y_1$. Then y_2 is primitive (since v_1 is primitive mod p), and $\text{Ann}(y_2)$ is an invariant ideal containing I_2 . We continue in this fashion until we reach an m such that $\text{Ann}(y_m) = I_m$. This must happen before we reach $n + 1$.

Now we construct M_i by induction, by applying this construction to M/M_{i-1} . Since M is finitely generated and B is Noetherian, M is a Noetherian module, so $M_s = M$ for some s . \square

6. Weak equivalences of Hopf algebroids

In this section, we show that our equivalences of comodule categories are induced by weak equivalences of Hopf algebroids.

Definition 6.1. Suppose $\Phi: (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. We say that Φ is a *weak equivalence* if the induced functor $\Phi_*: \Gamma\text{-comod} \rightarrow \Sigma\text{-comod}$, where $\Phi_* M = B \otimes_A M$, is an equivalence of categories.

We have the following characterization of weak equivalences of Hopf algebroids.

Theorem 6.2. A map $\Phi = (\Phi_0, \Phi_1): (A, \Gamma) \rightarrow (B, \Sigma)$ of flat Hopf algebroids is a weak equivalence if and only if the composite

$$A \xrightarrow{\eta_R} \Gamma \xrightarrow{\Phi_0 \otimes 1} B \otimes_A \Gamma$$

is a faithfully flat ring extension and the map

$$B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$$

that takes $b \otimes x \otimes b'$ to $\eta_L(b)\Phi_1(x)\eta_R(b')$ is a ring isomorphism.

One can rephrase this theorem using sheaves of groupoids. A Hopf algebroid (A, Γ) has an associated sheaf of groupoids $\text{Spec}(A, \Gamma)$ with respect to the flat topology on Aff , the opposite category of commutative rings (see [Hov02c]). Hollander [Hol01] has constructed a Quillen model structure on (pre)sheaves of groupoids in a Grothendieck topology, and Theorem 6.2 says that Φ is a weak equivalence if and only if $\text{Spec } \Phi$ is a weak equivalence of sheaves of groupoids.

Proof. The “if” half of this theorem is the main result of [Hov02c]. Conversely, suppose Φ is a weak equivalence. Then Φ_* is in particular exact, so that B is Landweber exact over A . Lemma 2.2 then guarantees that $B \otimes_A \Gamma$ is flat over A . On the other hand, if $B \otimes_A \Gamma \otimes_A M = 0$, then $\Phi_*(\Gamma \otimes_A M) = 0$, so, since Φ_* is an equivalence of categories, $\Gamma \otimes_A M = 0$. Since A is an A -module retract of Γ , we see that $M = 0$. Hence $B \otimes_A \Gamma$ is faithfully flat over A .

Now, if Φ_* is an equivalence of categories, then the counit $\Phi_*\Phi^*N \rightarrow N$ must be an isomorphism for all Σ -comodules N . In particular, $\Phi_*\Phi^*\Sigma \rightarrow \Sigma$ must be an isomorphism. But

$$\Phi_*\Phi^*\Sigma \cong B \otimes_A \Phi^*(\Sigma \otimes_B B) \cong B \otimes_A \Gamma \otimes_A B,$$

completing the proof. \square

For rings R and S , we can have equivalences of categories between R -modules and S -modules that are not induced by maps $R \rightarrow S$; this is, of course, the content of Morita theory. However, two commutative rings are Morita equivalent if and only if they are isomorphic. We view our Hopf algebroids as fundamentally commutative objects, so we do not expect any non-trivial Morita theory.

Conjecture 6.3. *Suppose (A, Γ) and (B, Σ) are flat Hopf algebroids such that the category of Γ -comodules is equivalent to the category of Σ -comodules. Then (A, Γ) and (B, Σ) are connected by a chain of weak equivalences.*

If this conjecture is going to be true, then in particular the equivalences of comodule categories in Theorem 4.2 must be induced by chains of weak equivalences. We will now prove this.

Let $(A, \Gamma) = (BP_*, BP_*BP)$ as usual. If we have two Landweber exact A -algebras B and B' of the same heights, and a map $(B, \Gamma_B) \rightarrow (B', \Gamma_{B'})$ under (A, Γ) , then it is a weak equivalence by Corollary 4.4. In general there may be no such map, though. We therefore record another obvious source of equivalences.

Suppose we have a groupoid with object set X and morphism set G . Given another set Y and a map $f: Y \rightarrow X$, we previously constructed a groupoid (Y, G_f) , where the morphisms in G_f from y_1 to y_0 are the morphisms in G from $f(y_1)$ to $f(y_0)$. Now suppose we have another map $g: Y \rightarrow X$, and thus another groupoid (Y, G_g) ; we want to know when this is equivalent to (Y, G_f) . Suppose we have a map $h: Y \rightarrow G$ such that $\text{target} \circ h = f$ and $\text{source} \circ h = g$, so that $h(y)$ is a morphism from $g(y)$ to $f(y)$ in G . We can then define a functor $H: G_g \rightarrow G_f$ by $H(y) = y$ on objects, and

$$H(y_0 \xleftarrow{u} y_1) = (f(y_0) \xleftarrow{h(y_0)} g(y_0) \xleftarrow{u} g(y_1) \xleftarrow{h(y_1)^{-1}} f(y_1))$$

on morphisms (for $u \in G_g(y_1, y_0) = G(g(y_1), g(y_0))$). Equivalently, let H' be the map

$$Y \times_{X, g} G \times_{X, g} Y \xrightarrow{h \times 1 \times (\text{invert} \circ h)} G \times_X G \times_X G \xrightarrow{\text{compose}} G.$$

Then $H(y_0, u, y_1) = (y_0, H'(y_0, u, y_1), y_1)$. It is easy to see that the functor H is an isomorphism of groupoids.

The analogue for Hopf algebroids is as follows.

Lemma 6.4. *Let (A, Γ) be a Hopf algebroid, and suppose $h: \Gamma \rightarrow B$ is a ring homomorphism. Let $f = h\eta_L$ and $g = h\eta_R$. Then there is an isomorphism of Hopf algebroids from (B, Γ_g) to (B, Γ_f) .*

Proof. The pair (A, Γ) represents a functor from graded rings to groupoids, and the conclusion follows from the above discussion by Yoneda's lemma. Alternatively, we can give a formula for the map $\Gamma_f \rightarrow \Gamma_g$ as follows. A map $\Gamma_f \rightarrow \Gamma_g$ of B -bimodules is equivalent to a map $\Gamma \rightarrow B \otimes_g \Gamma \otimes_g B$ of A -bimodules, where the target has the A -bimodule structure coming from f . This map is the composite

$$\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma \xrightarrow{\Delta \otimes 1} \Gamma \otimes_A \Gamma \otimes_A \Gamma \xrightarrow{h \otimes 1 \otimes (h \circ \chi)} B \otimes_g \Gamma \otimes_g B$$

(corresponding to H' in the previous discussion). \square

Theorem 6.5. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B and B' are Landweber exact A -algebras such that $\text{ht } B = \text{ht } B'$. Then the Hopf algebroids (B, Γ_B) and $(B', \Gamma_{B'})$ are connected by a chain of weak equivalences.*

Proof. Let $C = B \otimes_A \Gamma \otimes_A B'$. Let us denote C together with the ring homomorphism $f: A \rightarrow B \rightarrow C$ by C_f , and C together with the ring homomorphism $g: A \rightarrow B' \rightarrow C$ by C_g . Our desired chain of weak equivalences is

$$(B, \Gamma_B) \rightarrow (C_f, \Gamma_f) \cong (C_g, \Gamma_g) \leftarrow (B', \Gamma_{B'}).$$

The middle isomorphism comes from the evident map $h: \Gamma \rightarrow C$ such that $h\eta_L = f$ and $h\eta_R = g$ and Lemma 6.4.

We now claim that C_f , and therefore also C_g , is Landweber exact. Indeed, Lemma 2.2 implies that $B \otimes_A \Gamma$ and $B' \otimes_A \Gamma$ are flat over A . But then

$$C \otimes_A \Gamma = (B \otimes_A \Gamma) \otimes_A (B' \otimes_A \Gamma)$$

is also flat over A , and so C_f is Landweber exact.

Thus, it suffices to show that $\text{ht } C_f = \text{ht } C_g = \text{ht } B$. Because I_n is invariant, we have

$$C/I_n \cong (B/I_n) \otimes_A \Gamma \otimes_A (B'/I_n)$$

and therefore $B'/I_n = 0$ implies $C/I_n = 0$. Conversely, suppose $C/I_n = 0$, but $B'/I_n \neq 0$. This means that $\text{ht } B = \text{ht } B' \geq n$. Since

$$B \otimes_A (\Gamma \otimes_A B'/I_n) = 0,$$

we conclude that $\Gamma \otimes_A B'/I_n$ is $v_{\text{ht } B}$ -torsion, and therefore v_n -torsion. But B'/I_n is a retract of $\Gamma \otimes_A B'/I_n$ as an A -module, so B'/I_n is v_n -torsion. Since B' is Landweber exact, this means $B'/I_n = 0$, which is a contradiction. \square

7. The global case

The object of this section is to show that our results about Landweber exact algebras over BP_* extend to Landweber exact algebras over the complex cobordism ring MU_* . Recall that $MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$ for some generators x_i of degree $2i$. All we require of these generators is that the Chern numbers of x_{p^n-1} are all divisible by p , as in [Lan76]. In this case, the ideals $I_{p,n} = (p, x_{p-1}, \dots, x_{p^n-1})$ are invariant and independent of the choice of generators. These ideals and $I_{p,\infty} = \bigcup_n I_{p,n}$ are the only invariant prime ideals in MU_* [Lan76].

Our first goal is to understand the relation between graded hereditary torsion theories of MU_*MU -comodules and graded hereditary torsion theories of BP_*BP -comodules. We use the notation $A_{(p)}$ for $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and we recall the well-known fact that $(MU_*)_{(p)}$ is a Landweber exact BP_* -algebra of infinite height. Theorem 4.2 then gives us an equivalence of categories between graded $(MU_*MU)_{(p)}$ -comodules and graded BP_*BP -comodules.

Lemma 7.1. *Let \mathcal{T} be a graded proper hereditary torsion theory of graded MU_*MU -comodules, and, for a prime p , let $\mathcal{T}^{(p)}$ denote the class of p -torsion comodules in \mathcal{T} . Then $\mathcal{T} = \bigoplus_p \mathcal{T}^{(p)}$; that is, $M \in \mathcal{T}$ if and only if $M = \bigoplus M_{(p)}$ for $M_{(p)} \in \mathcal{T}^{(p)}$. Furthermore, there is a one-to-one correspondence between graded hereditary torsion theories of graded p -torsion MU_*MU -comodules and graded proper hereditary torsion theories of graded BP_*BP -comodules.*

Here we refer to a hereditary torsion theory as *proper* if it is not the entire category.

Proof. First of all, if \mathcal{T} is proper, then \mathcal{T} must consist entirely of comodules that are torsion as abelian groups. Indeed, suppose $M \in \mathcal{T}$ is non-torsion. Let $T(M)$ denote the torsion in M , which is easily seen to be a comodule. Let x be a nonzero primitive in $M/T(M) \in \mathcal{T}$. Then x is non-torsion. The annihilator ideal I of x is invariant, and we claim it is 0. Indeed, if I is nonzero, it must contain a nonzero invariant element of MU_* , which must be an integer m . But then $mx = 0$, contradicting the fact that x is non-torsion. The subcomodule of $M/T(M)$ generated by x is thus isomorphic to $s^t MU_*$ for some t , so $MU_* \in \mathcal{T}$. This implies that \mathcal{T} is the entire category of MU_* -comodules. Indeed, we then get $MU_*/I \in \mathcal{T}$ for all invariant ideals I . The Landweber filtration theorem implies that every finitely presented MU_*MU -comodule is in \mathcal{T} , and every comodule is a filtered colimit of finitely presented comodules.

Now it is easy to check that every torsion comodule M can be written as $\bigoplus_{(p)} M_{(p)}$, where $M_{(p)}$ is the p -localization of M and therefore is just the p -torsion in M . The correspondence between graded hereditary torsion theories of p -torsion MU_*MU -comodules and proper graded hereditary torsion theories of BP_*BP -comodules follows from the equivalence of categories between BP_*BP -comodules and $(MU_*MU)_{(p)}$ -comodules. \square

We then let $\mathcal{T}_n^{(p)}$ denote the hereditary torsion theory of p -torsion MU_*MU -comodules corresponding to \mathcal{T}_n . Thus $\mathcal{T}_n^{(p)}$ is generated by $MU_*/I_{p,n+1}$. For notational reasons, we let $\mathcal{T}_\infty^{(p)} = (0)$.

Definition 7.2. Given an MU_* -module B and a prime p , define the *height* of B at p , written $\text{ht}_p B$, to be the largest n such that $B/I_{p,n}$ is nonzero, or ∞ if $B/I_{p,n}$ is nonzero for all n .

We then have the integral analogue of Theorem 4.2.

Theorem 7.3. Let $(A, \Gamma) = (MU_*, MU_*MU)$, and suppose B and B' are two graded Landweber exact A -algebras with $\text{ht}_p B = \text{ht}_p B'$ for all primes p . Then the category of graded Γ_B -comodules is equivalent to the category of graded $\Gamma_{B'}$ -comodules, and both categories are equivalent to the localization of the category of graded Γ -comodules with respect to the torsion theory $\bigoplus_p \mathcal{T}_{\text{ht}_p B}^{(p)}$. This localization is the full subcategory of graded Γ -comodules consisting of all those M such that

$$\text{Hom}_A^*(A/I_{p, \text{ht}_p B+1}, M) = \text{Ext}_\Gamma^{1,*}(A/I_{p, \text{ht}_p B+1}, M) = 0$$

for all p such that $\text{ht}_p B < \infty$.

Note that this theorem implies Theorem 2.5, since if B is a Landweber exact BP_* -algebra, it is also a Landweber exact MU_* -algebra.

Proof of Theorem 7.3. Theorem 2.5 implies that graded Γ_B -comodules are equivalent to the localization of the category of graded Γ -comodules with respect to the kernel \mathcal{T} of the functor $M \mapsto B \otimes_A M$. Given a prime p , let $\mathcal{T}^{(p)}$ denote the collection of p -torsion comodules in \mathcal{T} . If B is zero, there is nothing to prove, so we can assume B is nonzero and therefore \mathcal{T} is proper. Lemma 7.1 then implies that we need only check that $\mathcal{T}^{(p)} = \mathcal{T}_{\text{ht}_p B}^{(p)}$.

Suppose first that $\text{ht}_p B = \infty$. Then we claim that $\mathcal{T}^{(p)} = (0)$. Indeed, suppose M is a nonzero comodule in $\mathcal{T}^{(p)}$. By choosing a primitive in M , we find that $A/I \in \mathcal{T}^{(p)}$ for some proper invariant ideal I in A such that $p^r \in I$ for some r . But then I is an invariant ideal in $A_{(p)}$. The equivalence of categories between graded $\Gamma_{(p)}$ -comodules and graded BP_*BP -comodules preserves invariant ideals. Since every proper invariant ideal in BP_* is contained in I_∞ , we see that $I \subseteq I_{p,\infty}$. Thus $A/I_{p,\infty} \in \mathcal{T}$. Hence $B/I_{p,\infty} = 0$. This means that $1 \in I_{p,\infty} B$, so $1 \in I_{p,n} B$ for some n . But then $B/I_{p,n} = 0$, violating our assumption that $\text{ht}_p B = \infty$.

Now suppose that $\text{ht}_p B = n < \infty$. Then $A/I_{p,n}$ is not in $\mathcal{T}^{(p)}$ but $A/I_{p,n+1} \in \mathcal{T}^{(p)}$. Propositions 3.2 and 3.3 imply that $\mathcal{T}^{(p)}$ corresponds to the hereditary torsion theory \mathcal{T}_n of BP_*BP -comodules.

The characterization of local objects follows from the fact that $\mathcal{T} = \bigoplus_p \mathcal{T}^{(p)}$, Lemma 1.8, and Corollary 4.3. \square

We then get analogues of the results of Sections 4–6 for MU_*MU -comodules. We will state only the structure theorem for comodules. We have the same difficulty with the classification of invariant prime ideals that we have with invariant radical ideals in the BP_* -case. We fix it analogously. That is, if \mathcal{A} is a symmetric monoidal abelian category with unit k for the tensor product, we define a categorical ideal I in \mathcal{A} to be *categorically prime* if $JK \leq I$ for categorical ideals J and K implies that $J \leq I$ or $K \leq I$. One checks that Landweber’s classification of invariant prime ideals in MU_* [Lan73] actually classifies categorically prime invariant ideals.

Theorem 7.4. *Let $(A, \Gamma) = (MU_*, MU_*MU)$, and suppose B is a Landweber exact A -algebra.*

- (a) *Every nonzero graded Γ_B -comodule has a nonzero primitive.*
- (b) *The categorically prime invariant ideals in B are $\{I_{p,n}B \mid 0 \leq n \leq \text{ht}_p B\}$. In particular, this set contains all the invariant prime ideals.*
- (c) *If B is Noetherian, then every graded Γ_B -comodule that is finitely generated over B admits a finite filtration by subcomodules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$$

for some s , with $M_r/M_{r-1} \cong s^l B/I_{p,j}$ for some s, p , and j depending on r with $j \leq \text{ht}_p B$.

In particular, this theorem applies to K_*K -comodules or Ell_*Ell -comodules, where K is complex K -theory, and Ell is one of the many versions of (periodic, complex oriented) elliptic cohomology.

We leave the proof of this theorem to the interested reader, except for a few comments that illustrate the differences between this theorem and Theorem D. First of all, in part (b) we need to assume that I is categorically prime, rather than just categorically radical. This is already true in MU_* , since the ideal (6), for example, is an invariant radical ideal in MU_* not of the desired form.

Secondly, in the proof of part (c), we need a Noetherian hypothesis that is not present in the corresponding fact for $(A, \Gamma) = (BP_*, BP_*BP)$. The reason for this is that, in the BP_*BP case, the category of Γ_B -comodules is either equivalent to the category of BP_*BP -comodules or to the category of $E(n)_*E(n)$ -comodules. In the first case, we already know the Landweber filtration theorem, and in the second case $E(n)_*$ is Noetherian. We believe that Theorem 7.4(c) is true without the Noetherian hypothesis, however.

8. BPJ_*BPJ -Comodules

Throughout this section, we let J be a fixed invariant sequence $p^{i_0}, v_1^{i_1}, \dots, v_{k-1}^{i_{k-1}}$ in BP_* of length k . The spectrum BPJ is constructed from BP by killing this regular sequence, as in [JY80], or, in a more modern fashion, in [EKMM97, Chapter V] or [Str99]. Then BPJ is an associative ring spectrum, with $BPJ_* = BP_*/J$. We will assume that the product on BPJ has been chosen to be commutative. This is always possible if $p > 2$, and we believe that it is possible for a cofinal set of ideals J when $p = 2$ although we have not checked the details. However, it is not possible when $p = 2$ and $J = I_k$.

The co-operation ring BPJ_*BPJ is not evenly graded if $k > 0$, but is still free over BPJ_* , so that (BPJ_*, BPJ_*BPJ) is a Hopf algebroid. (When BPJ is not commutative, the structure is more complicated.) The object of this section is to extend our results to Landweber exact BPJ_* -algebras B . The most important case is when $J = I_n$; the spectrum BPI_n is often called $P(n)$. The Morava K -theory coefficient ring $K(n)_*$ is Landweber exact over $P(n)_*$ [Yos76].

Our first job is to classify the hereditary torsion theories of BPJ_*BPJ -comodules. As before, we let \mathcal{T}_n denote the class of all graded BPJ_*BPJ -comodules that are v_n -torsion. By Lemma 2.3 of [JY80], M is v_n -torsion if and only if M is I_{n+1} -torsion. Of course, any BPJ_* -module is automatically I_k -torsion, so this is only interesting for $n \geq k - 1$.

The following theorem is our generalization of Theorem 3.1.

Theorem 8.1. *Let \mathcal{T} be a graded hereditary torsion theory of graded BPJ_*BPJ -comodules, and suppose that \mathcal{T} contains some nonzero comodule that is finitely presented over BPJ_* . Then $\mathcal{T} = \mathcal{T}_n$ for some $n \geq k - 1$.*

This theorem is proved just as Theorem 3.1, except that the results of [Lan79], which we used in the proof of Lemma 3.4, are not written so as to apply to BPJ_*BPJ -comodules. So one can either reprove the results of [Lan79] in this case, or construct a direct proof of Lemma 3.4 for BPJ_*BPJ -comodules using the results of [JY80].

We can then define the height for BPJ_* -algebras.

Definition 8.2. Suppose B is a nonzero graded BPJ_* -module. We define the *height* of B , written $\text{ht } B$, to be the largest n such that B/I_n is nonzero, or ∞ if B/I_n is nonzero for all n .

Note that every nonzero BPJ_* -module B has $\text{ht } B \geq k$.

Here is the analogue of Theorem 4.2.

Theorem 8.3. *Let $(A, \Gamma) = (BPJ_*, BPJ_*BPJ)$, and suppose B and B' are two graded Landweber exact BPJ_* -algebras with $k \leq \text{ht } B = \text{ht } B' = n \leq \infty$. Then the category of graded Γ_B -comodules is equivalent to the category of graded $\Gamma_{B'}$ -comodules. If $n = \infty$,*

these categories are equivalent to the category of graded Γ -comodules. If $n < \infty$, these categories are equivalent to the localization of the category of graded Γ -comodules with respect to the torsion theory \mathcal{T}_n .

We then get analogues of the results of Sections 4–6 for BPJ_*BPJ -comodules. These depend on the results of [JY80] on the structure of BPJ_*BPJ -comodules to replace the results of Landweber on the structure of BP_*BP -comodules.

In particular, we get versions of the Miller–Ravenel, Morava, and Hovey–Sadofsky change of rings theorems. These use the spectra $v_n^{-1}BPJ$ and $E(n, J)$ for $n \geq k$ in place of $v_n^{-1}BP$ and $E(n)$. Here $E(n, J)_*$ is Landweber exact over BPJ_* with

$$E(n, J)_* \cong v_n^{-1}(BPJ_*/(v_{n+1}, v_{n+2}, \dots)).$$

Here is the structure theorem for comodules.

Theorem 8.4. *Let $(A, \Gamma) = (BPJ_*, BPJ_*BPJ)$, and suppose B is a Landweber exact A -algebra.*

- (a) *Every nonzero graded Γ_B -comodule has a nonzero primitive.*
- (b) *The categorically radical invariant ideals in B are $\{I_n B \mid k \leq n \leq \text{ht } B\}$.*
- (c) *Every graded Γ_B -comodule that is finitely presented over B admits a finite filtration by subcomodules*

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_s = M$$

for some s , with $M_r/M_{r-1} \cong s^t B/I_j$ for some s, p , and j depending on r with $k \leq j \leq \text{ht } B$.

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